Induced Matching Partition of Certain Graphs

A. S. Shanthi
Department of Mathematics, Loyola College, Chennai 600 034, India
shanthu.a.s@gmail.com

Abstract- A matching \( M \) in a graph \( G = (V, E) \) is a set of mutually nonadjacent edges of \( G \). An induced matching \( k \)-partition of a graph \( G \) which has a perfect matching is a \( k \)-partition \( (V_1, V_2, \ldots, V_k) \) of \( V(G) \) such that, for each \( i \) \((1 \leq i \leq k)\), \( E(V_i) \) is an induced matching of \( G \) that covers \( V_i \). Determining induced matching \( k \)-partition number even when \( k = 2 \) is an \( \text{NP} \)-complete problem. In this paper we obtain the induced matching partition number of augmented butterfly and wrapped butterfly networks. Further we prove that the induced matching \( k \)-partition does not exist for Benes network.

Keywords- Matching; Induced Matching Partition; Augmented Butterfly Network; Wrapped Butterfly Network; Benes Network

I. INTRODUCTION

The prevalence of cloud computing is driving the deployment of data centres to host various network applications and services. In order to support computation and storage, demanding large scale distributed applications data centres have to accommodate a large number of interconnected servers. A typical data centre today has thousands of servers; a data centre in the future can have hundreds of thousands or even millions of servers. Although the availability of inexpensive commodity PCs has made it possible to expand a data centre to a huge number of servers, efficiently interconnecting the servers is still a challenging task [13].

Graph partitioning is a common pre-processing step in many high-performance parallel algorithms [19] and can be used to divide the work and data for an efficient parallel computation [7]. Graph partitioning is an important problem that has extensive applications in many areas, including scientific computing, VLSI design, task scheduling, geographical information systems, and operations research [10].

A matching \( M \) in a graph \( G = (V, E) \) is a set of mutually nonadjacent edges of \( G \). Perfect matching of a graph is a collection of independent edges, which together are incident on all the vertices of the graph. An induced matching \( k \)-partition of a graph \( G \) which has a perfect matching is a \( k \)-partition \( (V_1, V_2, \ldots, V_k) \) of \( V(G) \) such that, for each \( i \) \((1 \leq i \leq k)\), \( E(V_i) \) is an induced matching of \( G \) that covers \( V_i \), or equivalently, the subgraph \( G[V_i] \) of \( G \) induced by \( V_i \) is 1-regular. The induced matching partition number of a graph \( G \), denoted by \( \text{imp} \ (G) \), is the minimum integer \( k \) such that \( G \) has an induced matching \( k \)-partition.

Historically, the induced matching \( k \)-partition problem was first studied as an combinatorial optimization problem [6]. The induced matching \( k \)-partition problem is \( \text{NP} \)-complete, and also \( \text{NP} \)-complete even for \( k = 2 \) and for 3-regular planar graphs, respectively [6, 18]. Yuan and Wang have characterized graphs \( G \) with \( \text{imp} \ (G) = 2\Delta(G) - 1 \) where \( \Delta(G) \) is the maximum degree of \( G \). The induced matching problem was studied for certain interconnection networks such as butterfly networks, hypercubes, cube-connected cycles and grids [13]. The problem has been solved for honeycomb networks, honeycomb torus, sierpinski graphs and sierpinski gasket graphs [10]. In this paper we determine the induced matching partition number for augmented butterfly network and wrapped butterfly network. Further we prove that the induced matching \( k \)-partition does not exist for Benes network.

II. BENES NETWORK

A multistage network consists of a series of switch stages and interconnection patterns, which allows \( N \) inputs to be connected to \( N \) outputs. The butterfly and Benes networks are important multistage interconnection networks, which possess attractive topologies for communication networks. They have been used in parallel computing systems such as IBM SP1/SP2, MIT Transit Project, and NEC Cenju-3, and used as well in the internal structures of optical couplers, e.g., star couplers [12]. Butterfly graphs were originally defined as the underlying graphs of fast Fourier transform (FFT) networks, which can perform the FFT very efficiently [8].

The \( r \)-dimensional Benes network consists of back-to-back butterfly, denoted by \( BB \ (r) \). The \( BB \ (r) \) has \( 2r + 1 \) levels, each with \( 2^r \) vertices. The first and last \( r + 1 \) levels in the \( BB \ (r) \) form two \( BF \ (r) \)'s respectively, while the middle level in \( BB \ (r) \) is shared by these butterfly networks. The network shown in Fig. 1 is a 3-dimensional Benes network \( BB(3) \). Manuel et al. [13] showed that the induced matching partition of butterfly network \( BF \ (r) \), \( r \) odd was even and \( \text{imp} \ (BF \ (r)) \), \( r \) even did not exist. In this section we prove that the induced matching partition does not exist for Benes network.

Fig. 1 Benes Network \( BB \ (3) \)
Lemma 1 [16] Perfect matchings do not exist for a bipartite graph $G = (V, E)$ with $V = X \cup Y$, $|X| = m$, $|Y| = n$ and $m \neq n$.

Lemma 2 If a graph $G$ contains an induced subgraph $H$ isomorphic to $AB_2$ such that for the vertices at level 0 of $AB_2$, $\text{deg}_G u = \text{deg}_H u$, then $\text{imp}(G) \geq 3$.

Proof. We label vertices in $H$ as shown in Fig. 2. We claim that $\text{imp}(AB_2) \geq 3$. Suppose on the contrary that $V_1, V_2$ form an induced matching 2-partition of $H$. Label vertices in $V_1$ as 1 and $V_2$ as 2. Consider $(1, 00), (1, 01), (1, 10), (1, 11)$. Without loss of generality suppose $(1, 00), (1, 01) \in V_1$ and $(1, 10), (1, 11) \in V_2$ then $[0, 00](0, 10) \in V_1 \cup V_2$. If $(1, 00), (1, 10) \in V_1$ and $(1, 01), (1, 11) \in V_2$ then $[2, 00](2, 01) \notin V_1 \cup V_2$. If $(1, 00), (1, 11) \in V_1$ and $(1, 01), (1, 10) \in V_2$ then either $(0, 01)$ or $(0, 11)$ or $(2, 00)$ or $(2, 01) \notin V_1 \cup V_2$. The labelling shown in Fig. 3 implies $\text{imp}(AB_2) = 3$. In $G$, vertex $v$ cannot receive label 1 or 2. Hence $\text{imp}(G) \geq 3$.

Lemma 3 Perfect matchings do not exist for a bipartite graph $G = (V, E)$ with $V = X \cup Y$, $|X| = m$, $|Y| = n$ and $m \neq n$.

Proof. Let us assume that the result is true for augmented butterfly network $AB_2$, $n \geq 2$. Then $\text{imp}(AB_n) = 3$. We prove the result by induction on the dimension of the augmented butterfly network. For $k = 2$, let us label the vertices of $AB_2$ as in Fig. 3. $AB_3$ consists of two copies of augmented butterfly network of dimension 2 say $AB_2'$ and $AB_2''$. $AB_2'$ is labeled as shown in the Fig. 3 and $AB_2''$ is labeled as the complement of $AB_2'$ with 1 being labeled as 2, 2 as 3 and 3 as 1. The 0 level vertices in $AB_3$ are labeled as 2. See Fig. 4. Let us assume that the result is true for augmented butterfly network of dimension $n$. Now $AB_{n+1}$ contains two copies of $AB_n'$ and $AB_n''$. By induction hypothesis $AB_n'$ is labeled as $AB_n$ and $AB_n''$ is labeled as the complement of $AB_n'$ with 1 being labeled as 2, 2 as 3 and 3 as 1. The zero level vertices in $AB_{n+1}$ are labeled as 1 or 2 or 3 according as $(1, x_1 x_2 x_3 \ldots x_n)$ is labeled as 2 and 3 or 1 and 2. Since the label of $AB_2$ form an induced matching partition, by induction hypothesis $AB_n'$ and $AB_n''$ form an induced matching partition. Since $AB_n'$ and $AB_n''$ are complement to each other $(1, x_1 x_2 x_3 \ldots x_n)$ and $(1, x_1 x_2 x_3 \ldots x_n)$ receive different labels. By construction $(0, x_1 x_2 x_3 \ldots x_n)$ receives label different from $(1, x_1 x_2 x_3 \ldots x_n)$ and $(1, x_1 x_2 x_3 \ldots x_n)$. Therefore $\text{imp}(AB_n) = 3$. □

III. AUGMENTED BUTTERFLY NETWORK

The butterfly network $BF_n$ is one of the most widely used interconnection networks, very efficiently used in parallel computers and ATM switches. Also its properties are widely studied in the literature [1, 4]. But the butterfly network has many weaknesses too. It is non-Hamiltonian, not pancyclic and its toughness is less than one. Further not many variations of the butterfly network are available. In order to improve the properties of butterfly network, Manuel et al. introduced augmented butterfly network [14].

Definition 1 [14] Let $n \geq 1$ be an integer. The vertices of the n-dimensional augmented butterfly network are the pairs $(r, x)$ where $r$ is a non-negative integer $0 \leq r \leq n$ called the level and $x = (x_1 x_2 x_3 \ldots x_n)$ is a binary string of length $n$. In $AB_n$ the vertex $(r, x)$, $0 \leq r \leq n - 1$, is adjacent to the vertices $(r + 1, x)$, $(r + 1, x_1 x_2 x_3 \ldots x_r x_{r + 1} x_{r + 2} \ldots x_n)$, $(r, x_1 x_2 x_3 \ldots x_r x_{r + 1})$, and $(r, x_1 x_2 x_3 \ldots x_r x_{r + 2} \ldots x_n)$. Further the vertex $(n, x_1 x_2 x_3 \ldots x_n)$ is adjacent to $(n, x_1 x_2 x_3 \ldots x_n)$. See Fig. 2.

![Fig. 2 Augmented butterfly network of dimension 2](image-url)

In particular, when $r = 0$, the vertex $(0, x_1 x_2 x_3 \ldots x_n)$ is adjacent to the vertices $(1, x_1 x_2 x_3 \ldots x_n)$, $(1, x_1 x_2 x_3 \ldots x_n)$ and $(0, x_1 x_2 x_3 \ldots x_n)$. Also when $r = n$, $(n, x_1 x_2 x_3 \ldots x_n)$ is adjacent to the vertices $(n, x_1 x_2 x_3 \ldots x_n)$, $(n - 1, x_1 x_2 x_3 \ldots x_n)$ and $(n, x_1 x_2 x_3 \ldots x_n)$. Clearly $AB_n$ has $(n + 1) 2^n$ vertices and $3n \times 2^n$ edges.

Remark 1 [14] The edges between $(r, x)$ and $(r, x_1 x_2 x_3 \ldots x_r x_{r + 1} x_{r + 2} \ldots x_n)$ and between $(r, x)$ and $(r, x_1 x_2 x_3 \ldots x_r x_{r + 1} x_{r + 2} \ldots x_n)$ are called level edges. The edges between $(r, x)$ and $(r + 1, x)$ are called straight edges while the edges between $(r, x)$ and $(r + 1, x_1 x_2 x_3 \ldots x_r x_{r + 1} x_{r + 2} \ldots x_n)$ are called cross edges.

![Fig. 3 Induced Matching Partition of $AB_2$](image-url)

Theorem 2 Let $G$ be the augmented butterfly network $AB_n$, $n \geq 2$. Then $\text{imp}(G) = 3$.

Proof. $AB_n$ contains $AB_2$ as an induced subgraph satisfying the condition of Lemma 2. Therefore $\text{imp}(AB_n) \geq 3$. We now proceed to prove that $\text{imp}(AB_n) = 3$. We prove the result by induction on the dimension of the augmented butterfly network. For $k = 2$, let us label the vertices of $AB_2$ as in Fig. 3. $AB_3$ consists of two copies of augmented butterfly network of dimension 2 say $AB_2'$ and $AB_2''$. $AB_2'$ is labeled as shown in the Fig. 3 and $AB_2''$ is labeled as the complement of $AB_2'$ with 1 being labeled as 2, 2 as 3 and 3 as 1. The 0 level vertices in $AB_3$ are labeled as 2. See Fig. 4. Let us assume that the result is true for augmented butterfly network of dimension $n$. Now $AB_{n+1}$ contains two copies of $AB_n'$ and $AB_n''$. By induction hypothesis $AB_n'$ is labeled as $AB_n$ and $AB_n''$ is labeled as the complement of $AB_n'$ with 1 being labeled as 2, 2 as 3 and 3 as 1. The zero level vertices in $AB_{n+1}$ are labeled as 1 or 2 or 3 according as $(1, x_1 x_2 x_3 \ldots x_n)$ is labeled as 2 and 3 or 1 and 2. Since the label of $AB_2$ form an induced matching partition, by induction hypothesis $AB_n'$ and $AB_n''$ form an induced matching partition. Since $AB_n'$ and $AB_n''$ are complement to each other $(1, x_1 x_2 x_3 \ldots x_n)$ and $(1, x_1 x_2 x_3 \ldots x_n)$ receive different labels. By construction $(0, x_1 x_2 x_3 \ldots x_n)$ receives label different from $(1, x_1 x_2 x_3 \ldots x_n)$ and $(1, x_1 x_2 x_3 \ldots x_n)$. Therefore $\text{imp}(AB_n) = 3$. □
IV. WRAPPED BUTTERFLY NETWORK

The performance of a distributed system is significantly determined by its network topology. Though the hypercube (binary n-cube) is one of the most popular interconnection networks and has been used to design various commercial multiprocessor machines, one basic drawback with hypercubes is that the vertex degree increases with the number of vertices. On the other hand among all networks with fixed degrees, the wrapped butterfly network is one of the most promising networks due to its nice topological properties [21]. The set \( V \) of nodes of an \( r \)-dimensional butterfly correspond to pairs \([x, i]\), where \( 0 \leq i \leq r \), is the level of a vertices and \( x \) is an \( r \)-bit binary number that denotes the row of the vertices. Two vertices \([x, i]\) and \([y, i']\) are linked by an edge if and only if \( i' = i + 1 \) and either:

1. \( x \) and \( y \) are identical, or
2. \( x \) and \( y \) differ in precisely the \( i' \)th bit.

The edges in the network are undirected. An \( r \)-dimensional butterfly is denoted by \( BF (r) \). For fixed \( i \), the vertex \((x, i)\) is a vertex on level \( i \). See Fig. 5 (a). The wrapped butterfly, denoted by \( WBF (r) \) is obtained by merging the first and last levels of \( BF (r) \) into a single level. Thus vertex \((x, 0)\) is merged into vertex \((x, r)\) for each \( x \). The wrapped butterfly \( WBF (r) \) is an \( r \)-level graph with \( 2^r \) vertices, each of degree 4. The edges between \((x, i)\) and \((x, i + 1)\) are called straight edges while the edges between \((x, i)\) and \((x_1, x_2, \ldots, x_r, i + 1)\) are called cross edges and the edges between \((x, r)\) and \((x, 0)\) are called wrapped edges. See Fig. 5 (b).

\[ WBF (3) \]

(a)

(b)

\[ BF (3) \]

Theorem 3 Let \( G \) be the wrapped butterfly network \( WBF (r) \), \( r \) even. Then \( \text{imp} (G) = 2 \).

Proof. We prove the result by the method of induction on \( r \). Label vertices in \( V_1 \) as 1 and \( V_2 \) as 2. For \( r = 2 \), let us label the vertices of \( WBF (2) \) as shown in the Fig. 6. Clearly \( \text{imp} (WBF (2)) = 2 \). Assume that the result is true for wrapped butterfly of dimension \( r - 1 \). Consider \( BF (r) \). The removal of wrapped edges from \( WBF (r) \) leaves us two copies of butterfly network of dimension \( r - 1 \) say \( BF' (r - 1) \) and \( BF'' (r - 1) \). Now the vertices of \( BF'' (r - 1) \) are labeled as follows:

(i) The vertices \((00x_{r-1}, \ldots, x_{r-1}, i)\) are labeled as \((x_{r-1}, \ldots, x_{r-1}, i - 2)\) in \( WBF (r - 2) \), \( 3 \leq i \leq r \).

(ii) The vertices \((01x_{r-1}, \ldots, x_{r-1}, i)\) are labeled as the complement to the label of the vertices \((00x_{r-1}, \ldots, x_{r-1}, i)\), \( 3 \leq i \leq r \) where complement of label 1 is 2 and conversly.

(iii) The vertices \((0x_{r-1}, \ldots, x_{r-1}, 2)\) are labeled as the complement to the label of the vertices \((0x_{r-1}, \ldots, x_{r-1}, 3)\).

(iv) The vertices \((0x_{r-1}, \ldots, x_{r-1}, 1)\) are labeled as the complement to the label of the vertices \((0x_{r-1}, \ldots, x_{r-1}, 2)\).

Now the vertices of \( BF'' (r - 1) \) are labeled as the vertices of \( BF' (r - 1) \) where the vertex \((1x_1, x_2, \ldots, x_r, i)\) are identified with the vertex \((0x_1, x_2, \ldots, x_r, i)\), \( 1 \leq i \leq r \). It is enough to prove

\[ WBF (2) \]
The induced matching partition of wrapped butterfly network \( WBF(3) \) with \( \text{imp} = 3 \) is shown in Fig. 9. Further it is easy to check manually that \( \text{imp}(WBF(5)) = 3 \).

**Open Problem:** Let \( G \) be the wrapped butterfly network \( WBF(r) \), \( r \) odd. Then \( \text{imp}(G) = 3 \).

**Theorem 4** Let \( G \) be the wrapped butterfly network \( WBF(r) \), \( r \) odd. Then \( \text{imp}(G) \geq 3 \).

**Proof.** Suppose not. Let \( \text{imp}(G) = 2 \) and let \( V_1 \) and \( V_2 \) form an induced matching partition. Every vertex in \( WBF(r) \) lies in a cycle of length 4 say \( C: v_1v_2v_3v_4 \). Then there are two possibilities either \( v_1, v_2 \in V_1 \) or \( v_1, v_2 \in V_2 \) or \( v_1, v_2 \in V_1 \) and \( v_3, v_4 \in V_2 \). If \( WBF(r) \) every vertex lies in exactly two different cycles \( C \) and \( C' \) of length 4 and \( \deg x = 4 \) \( \forall x \in V \) \((WBF(r))\). If the vertices in \( C': pqrs \) receive the Labels 1, 1, 2, 2 then the vertices in the cycle \( C'' \) starting with the vertex \( p \) receive the Labels 1, 2, 1, 2. Hence in the cycles \( [(x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 2)] \), \( [(x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 2)] \), \( [(x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 1), (x_1x_2x_3...x_r, r – 2)] \), alternate cycles should receive the label 1, 1, 2, 2 and the remaining cycles receive Label 1, 2, 1, 2 and no two cycles receive the same label, but they are odd in number, which is not possible. See Fig. 8. Thus \( \text{imp}(G) \geq 3 \).

**V. CONCLUSION**

In this paper, the induced matching partition number of augmented butterfly network and wrapped butterfly network has been determined. As the induced matching \( k \)-partition problem is \( NP \)-complete even for \( k = 2 \), it would be interesting to identify other interconnection networks for which \( k = 2 \). It is also worth investigating interconnection networks for which \( k > 2 \).

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**REFERENCES**


