Tuning a Fractional Order PD and PID Controller with Lead Compensator for Integrating Time Delay Systems

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Abstract—In this paper two fractional order controllers with three adjustable parameters are investigated for first order integrating time delay systems. In order to tune controllers parameters at first stability boundary criteria is calculated. Stability boundary locus method is used for parameters tuning. Tuned parameters are not able to achieve demanded frequency characteristics such as gain and phase margins. For this reason a lead compensator should be designed. Bode diagrams which are drawn in frequency domain show attaining to favourite specifications. Numerical examples demonstrate accomplishment of these controllers with lead compensator.

Keywords- FO-PD; FO-PID; Lead compensator; Integrating time delay system; Stability boundary locus

I. INTRODUCTION

Integrating systems with time delay as a model of industrial processes has attracted a wide range of research efforts in the recent years [1]. In another view, for many physical processes, dynamic modelling may be done more accurately via delayed fractional differential equations involving non integer order derivatives [2-4]. Therefore an increasing number of studies related with the application of fractional calculus in many areas of science and engineering (see, e.g., [1], [5]). Using the differentiation and integration of fractional order (FO) or non-integer order in systems control is gaining more and more interests from the systems control community [6]. In what concerns automatic control theory the fractional calculus concepts are adapted to frequency domain based methods.

The frequency and the transient responses of the non-integer integral and its application to control systems were introduced by Manabe (see [7]) [8]. In theory, the control systems can include both the fractional order dynamic system or plant that to be controlled and the fractional-order controller. However, in control practice, more common is to consider the fractional order controller. This is due to the fact that the plant model may have already been obtained as an integer order model in classical sense and the objective is to apply the fractional order control to enhance the system control performance [6]. Generally, fractional order controllers divided into 4 classes: TID¹ controller, CRONE controller ², P²D³ controller and fractional lead-lag compensator [6].

The purpose of fractional order controller design is determined controller parameters so that the closed-loop system was stable and has optimal performance. There are different ways to designing this class of controllers in the both of time domain and frequency domain that each has own advantages and disadvantages. For instance in [9] a fractional differentiator by using system linearization in frequency domain for chaos control is designed. In [10] differentiators are designed in the time domain with using least square method. In [11] based on definition of piecewise orthogonal functions the parameters for fractional order PID (FO-PID)³ controller are determined. In [12] applying an improved differential evolution method a P²D³ controller is designed using discretization.

In [13-18] fractional order PI (FO-PI)⁴, fractional order PD (FO-PD)⁵ and FO-PID controllers are designed by using gain margin, cross over frequency and phase margin criteria in frequency domain for integer and fractional order systems. The first goal of this paper is using stability boundary locus method to design a class of fractional order PID controller with three adjustable parameters in frequency domain. This method has three conditions in frequency domain to design fractional order controllers and so for a controller with three adjustable parameters can be used well. On the other hand for better analysis and design of fractional order controllers a gain-phase margin tester can be used. Based on definition 2.3 in [19] this tester can be thought of as a "virtual compensator". The second goal is applying lead compensator to the controlled system. To the best knowledge, there is no method available to achieve gain and phase margins exactly.

However, phase-lead compensators have some parameters in the denominator of its transfer function, unlike PID controllers where all the parameters appear linearly. Thus, effective techniques for PID controllers with exact gain and phase margin specifications are not applicable to phase-lead compensators [20]. In industrial process control applications, phase-lead/lag compensators are widely used next only to PID controllers [21].

Such controllers are tuned usually with specifications on gain and phase margins which can lead to good performance and robustness [20]. In the tuning of phase lead/lag compensators, knowledge of specific points on the frequency response of the plant is required. Such points are specialized by their frequency, gain and phase and are not readily available without an accurate model of the plant [22].

The traditional tuning method for phase-lead/lag compensator parameters is based on the “trial and error” procedure Attempts towards analytical synthesis have been...
made since Wakeland in 1976 first proposed a one-step design for phase-lead compensators [21,23]. The major difficulties for tuning of phase-lead compensators under the gain and phase margin specifications lie in nonlinearity and coupling of all their three parameters [20]. Later on, Yeung, Chaid, and Dinh in 1995 developed a series of Bode design charts to allow “non-trial and error” designs of both continuous-time and discrete-time compensators [20]. Wang in 2003 presented the exact and unique solution to the design of phase-lead and phase-lag compensation when the desired gains in magnitude and phase are known at a given frequency [24]. Recently, in [20] another tuning method for phase lead compensators was proposed which can achieve the desired gain and phase margins exactly regardless of the plant order, time delay or damping nature.

In [21] a new approach is presented to first determine the stabilizing parameter set of phase-lead/lag compensators for all pole stable plants with time-delay and then to synthesize phase-lead/lag compensators with desired gain and phase margins. In [20] a simple and effective tuning method for phase-lead compensators which can achieve exact gain and phase margins simultaneously is presented. The third goal of this paper is comparing operation of FO-PD and FO-PID controller with and without lead compensator. For the reason presented in [17, 18], Here using the stability boundary locus in the frequency domain parameters of FO-PID controller are set. Gain-phase margin tester provides information for plotting the boundaries of constant gain margin and phase margin in the parameter plane only are calculated for FO-PID controller and the calculations of lead compensator for FO-PD controller are relinquished [19].

II. CONTROLLER TUNING PROCEDURE

Generally an integer order integrating time delay system described by the following dynamic equation:

\[ G(s) = \frac{K}{a_ns^n + \cdots + a_1s + a_0} e^{-\theta s} = \frac{K}{\sum_{i=0}^{n} a_is^i} e^{-\theta s} \]  

Where \( K \) is open-loop gain and \( a_i, (i = 0, \ldots, n) \) transfer function denominator coefficients and \( \theta \) the delay for the system.

A FO-PD controller with three tunable parameters is considered as follow [25]:

\[ C(s) = K_p + K_p s^{-\lambda} s^{\mu} \]  

In this section for briefness all calculations are done only for FO-PID controller. A class of fractional order controllers with three adjustable parameters is considered as follow:

\[ C(s) = K_p + K_p s^{-\lambda} - s^{\mu} \]  

Where \( K_p, K_p \) are the gains of controller and \( \mu, \lambda = 1 - \mu \in (0,1) \) are fractional order. This is one of the FO-PID controllers that they are not sensitive to the system parameters changing and due to fractional order having more flexibility in systems control [26].

As it said for better tuning of controller parameters a gain phase margin taster has been used that described as follow:

\[ C_f(A, \phi) = A e^{-j\phi} \]  

Where \( A \) is the gain margin and \( \phi \) is the phase margin. In order to finding controller parameters for a given value of gain margin \( A \) of the control system, one need to set \( \phi = 0 \) in Eq.(4).

On the other hand, setting \( A=1 \) in Eq.(4), one can obtain the controller parameters for a given phase margin \( \phi \). This tester in frequency domain is equal to a lead compensator. Generally a compensator is describing as below [26]:

\[ C_f(s) = \frac{K_1(s + z)}{(s + p)} \]  

Where \( K_1 \) is compensator’s gain and \( z, p \) are compensator’s zero and pole, respectively. So closed loop system equation is as follow:

\[ y = \frac{G(s)C(s)C_f(s)}{1 + G(s)C(s)C_f(s)}r \]  

The characteristic equation of closed loop system is as follow:

\[ P(s) = 1 + G(s)C(s)C_f(s) \]  

Based on Eq.(5) the compensator designing is related to choosing the \( z, p \) and \( K_1 \) in order to obtaining good performance.

If in Eq.(5) \( |z| < |p| \) then the compensator is a lead compensator. If the zero being very small (i.e. \( |p| >> |z| \)) or zero being at the origin then the compensator is equal to a derivative that described as bellow:

\[ C_f(s) \approx \frac{K_1 s}{p} \]  

This derivative has the following frequency response:

\[ C_f(j\omega) = jK_1 \frac{\omega}{p} = K_1 e^{j90^\circ} \]  

So for Eq.(5), the next equation will be arrived:

\[ C_f(j\omega) = \frac{K_1(z/p)(j\omega/z + 1)}{[j(\omega/p) + 1]} \approx \frac{K_2(1+j\alpha z)}{(1+j\alpha \tau)} \]  

Where \( p = \alpha z, \tau = 1/p \) and \( K_2 = K_1/\alpha \).

Therefore the transfer function for lead compensator is as follow:

\[ C_f(s) = K_2(1+\alpha z s) \approx \frac{K_2(1+\alpha z s)}{(1+z s)} \]  

\[ \phi(\omega) = \tan^{-1} \alpha \omega \tau - \tan^{-1} \omega \tau \]  

The maximum lead occurs on \( \omega_m \) that it is the through geometric of \( p=1/\tau \) and \( z=1/\alpha \). It means that the maximum lead on logarithm frequency axis is in the center of zero and pole frequencies, then

\[ \omega_m = \sqrt{\frac{p}{\tau}} = \frac{1}{\tau \sqrt{\alpha}} \]
In order to obtaining the maximum angle the Ed.(11) has been rewritten as follow:

$$\phi = \tan^{-1} \left( \frac{-\alpha \omega - \omega \tau}{1 + (\omega \tau)^2 \alpha} \right)$$

(14)

With substituting \(\omega_m = 1/\tau \sqrt{\alpha} \) in Eq.(14) the next equation is obtained:

$$\tan \phi_m = \left( \frac{\alpha \sqrt{\tau}}{1 - (1/\sqrt{\alpha})} \right) = \frac{-\alpha}{1 + 1}$$

(15)

For a given FO-PID controller parameters \(K_p, K_i, \lambda, \mu\) the closed-loop system is said to be bounded-input bounded-output (BIBO) stable if the quasi-polynomial \(P(s)\) has no roots in the closed right-half of the s-plane (RHP). The stability domain \(S\) is the parameter space \(\mathcal{P}\) with \(K_p, K_i, \lambda, \mu\) being coordinates is the region that for \((K_p, K_i, \lambda, \mu) \in S\) the roots of quasi-polynomial \(P(s)\) all lie in open left-half of the s-plane (LHP).

The boundaries of the stability domain \(S\) which are described by real root boundary (RRB), infinite root boundary (IRB) and complex root boundary (CRB) can be determined by the D-decomposition method [28, 29]. These boundaries are defined by equations \(P(0, L) = 0\), \(P(\infty, L) = 0\) and \(P(\pm j \omega, L) = 0\) for \(\omega \in (0, \infty)\), respectively, where \(P(s, L)\) is the characteristic function of the closed loop system and \(L\) is the vector of controller parameters. In applying the descriptions of stability boundaries of the stability domain \(S\) to the FO-PID in Eq.(7), the RRB turns out to be simply a straight line given by:

\[ KK_p K_p = 0 \rightarrow K_p = 0 \]  

(16)

To construct the CRB, \(j \omega = \infty\) has been substituting into Eq.(7) and the following equation has been obtained:

\[ P(j \omega) = 1 + \left( 1 + j \alpha \omega \right)^{-1}(K_p + K_i (j \omega))^{-1}(1 + j \omega \sum a_i (j \omega)^{i}) \]

(17)

For given gain and phase margin, \(K_p, K_i\) is obtained as follow:

\[ K_p = \sum a_i c_i \left( F(\omega) + \tau D(\omega) \right) - \frac{KK_p(A(\omega)F(\omega) - B(\omega)D(\omega))}{K_p \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right) + K_i \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right)} \]

(18)

\[ K_i = \frac{\sum a_i d_i \left( D(\omega) - \tau D^2(\omega) \right) + K \left( F(\omega)C(\omega) - D(\omega)E(\omega) \right)}{KK_p \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right) + K_i \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right)} + \frac{a_i c_i \left( F(\omega) + \tau D(\omega) \right)}{KK_p \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right) + K_i \left( A(\omega)F(\omega) - B(\omega)D(\omega) \right)} \]

(19)

Where

\[ c_i = \text{Re}(j \omega)^i \quad d_i = \text{Im}(j \omega)^i \]

(20)

\[ A(\omega) = \cos(\omega \theta) + \alpha \text{cos}(\omega \theta), B(\omega) = \sin(\omega \theta) - \alpha \text{cos}(\omega \theta) \]

(21)

\[ [C(\omega) = \omega^{-1}(A(\omega)\cos(\frac{\mu}{2}) + B(\omega)\sin(\frac{\mu}{2})) \]

\[ D(\omega) = \omega^{-1}(A(\omega)\cos(\frac{\mu}{2}) - B(\omega)\sin(\frac{\mu}{2})) \]

\[ E(\omega) = \omega^{-1}(B(\omega)\cos(\frac{\mu}{2}) - A(\omega)\sin(\frac{\mu}{2})) \]

\[ F(\omega) = \omega^{-1}(B(\omega)\cos(\frac{\mu}{2}) + A(\omega)\sin(\frac{\mu}{2})) \]

(22)

In order to plotting stability boundary locus \(\omega\) should be changed from 0 to \(\infty\). For \(A = 1\) and \(\phi = 0\), the stability boundaries RRB, IRB and CRB divide the parameter \((K_p, K_i)\) into stable and unstable regions. The stable region can be found by checking one arbitrary test point within each region. The characteristic equation belonging to the stable region has no RHP roots while the characteristic equation of the unstable region has a certain number of RHP roots. For checking the stability of the fractional-order characteristic equation, an effective numerical algorithm is given in [3].

The region having the stable characteristic equation, which is called the “general stability region”, gives a set of the stabilizing \(K_p\) and \(K_i\) parameters for the fixed values of \(\mu\) and \(\lambda\). It is noted that different choice of \(\mu\) and \(\lambda\) lead to different general stability regions [19].

In the next examples first for the given transfer functions general stability region in parameters space \((K_p, K_i)\) is plotted and the good performance of the closed loop system that controlled with FO-PID and lead compensator is obtained. Then the stability region for the transfer function which is controlled with FO-PD is shown and at the end of examples section the performance for closed loop system with FO-PD and lead compensator will be presented.

III. ILLUSTRATIVE EXAMPLES

A. Example 1

The first order integrating time delay system considered as follow:

\[ G(s) = \frac{K}{a_i s} e^{-\theta s} \]

(23)

Where \(K = 1, a_i = 5\) and \(\theta = 1\). At first fractional order PID controller with three tuneable parameters by using of frequency domain specification has been designed. Considering \(\mu\) and \(\lambda\) is 0.5 and \(\omega\) is changing from 0 to 11.5. So the stable and unstable regions without using lead compensator are as Fig.1. It is seen that from Fig.1 the shaded region (R3) is the stability region for closed loop system. In order to better vision it is shown in Fig.2. Actually it is stability boundary region.

Based on Fig.2 it is seen that the maximum of \(\phi\) in the stability boundary region for closed loop system without using lead compensator is 3.9773. The accuracy of the found
stability region can be easily tested using the unit step responses of the closed loop system.

Based on Eq.(18) and Eq.(19), the stability region for the $PI^{0.5} D^{0.5}$ controller with using of lead compensator is shown in Fig. 4. The shaded region ($R_3$) is the stability region or the stability boundary locus. As seen in this figure, the stability region is extended from $\omega = 0$ up to $\omega = 5.3698$ and so this region is bigger than the region that is shown in Fig.2. Therefore, it is true to say that using lead compensator not only improved the gain and phase margin but also increased the stability boundary locus for control system. In order to compensator performance review, the bode diagram for open loop system without and with using of lead compensator are plotted. Fig.6 shows bode diagram for control system by $PI^{0.5} D^{0.5}$ without using lead compensator.

In this paper, fractional-order operators have been approximated by continued fraction expansion of the direct discretization by recursive Tustin transformation [13]. The unit step responses of the $PI^{0.5} D^{0.5}$ control system when $K_f$ is chosen as 0.5 and various values of $K_p$ is shown in Fig. 3. In this figure, $K_p$ value has been chosen 1, 3, 6 and 8.125. From this figure, it is seen that the control system has more oscillatory response when the value of $K_p$ is increased from 0 to boundary value, $K_p = 8.125$. If the $K_p$ is bigger than the boundary value or smaller than zero, the control system becomes unstable. Now, for this system the gain margin and phase margin has been considered 11 and 80 respectively. So the gain-phase margin tester has $A = 1, \phi = 80^\circ$.

According to this figure the gain margin is 10.5 dB and phase margin is 68 degree. To obtaining desired values of gain-phase margin tester it is necessary the system phase margin increased to 80 degree. Doing the procedures provided in section 2 for designing lead compensator, control system was compensated has the bode diagram that shown in Fig.7. It can be seen from this figure that the phase margin in favourite frequency has reached to desired phase margin (i.e. 80 degree) and the system gain has not changed. Step responses of closed loop system by using of lead compensator are shown in Fig.8. From this figure it is seen that the maximum value $K_p$ for
compensated system somewhat increased \( K_p = 8.34 \). Thus the good performance of closed loop system is obtained.

In order to better vision this region is shown in Fig.10. As it said in fact it is stability boundary region. Based on Fig.10 it is seen that the maximum of \( \omega \) in the stability boundary region for closed loop system without using lead compensator is 2.36. Similar to Example 1, Unit step responses of the \( PD^{0.5} \) control system when \( K_D \) is chosen as 0.5 and various values of \( K_p \) is shown in Fig. 11.

**B. Example 2**

Consider the first order integrating time delay system with Eq.(23) from previous example. At first FO-PD controller with three tunable parameters by using of frequency domain specification has been designed. Considering \( \mu \) is 0.5 and \( \omega \) is changing from 0 to 11.5. So the stable and unstable regions without using lead compensator are as Fig.9. From Fig.9 it is seen that the shaded region \( (R3) \) is the stability region for closed loop system that controlled by FO-PD without lead compensator.
In Fig. 11, $K_P$ value has been chosen 0.5, 2.5, 5.5 and 7.668. From this figure, it is seen that the control system has more oscillatory response when the value of $K_P$ is increased from 0 to boundary value, $K_P = 7.668$. If the $K_P$ is bigger than the boundary value or smaller than zero, the control system becomes unstable.

Like the previous example gain margin and phase margin has been considered 11 and 80 respectively. So the gain-phase margin tester has $A = 1, \phi = 80^\circ$. Based on Eq.(17) and Eq.(18), $K_P$ and $K_D$ has been computed again. The stability region for the $PD^{0.5}$ controller with using of lead compensator is shown in Fig. 12.

The shaded region ($R_i$) is the stability region or the stability boundary locus. The stability boundary locus for FO-PD has been highlighted in Fig. 13 for better clearance.

As seen in Fig. 13, the stability region is extended from $\omega = 0$ up to $\omega = 8.1348$ and so this region is bigger than the region that is shown in Fig.10.

Therefore, it is true to say that using lead compensator not only improved the gain and phase margin but also increased the stability boundary locus for closed loop system. In order to compensator performance review, the bode diagram for open loop system without and with using of lead compensator are plotted. Fig.14 shows bode diagram for control system by $PD^{0.5}$ without using lead compensator. According to this figure the gain margin is 11.9 dB and phase margin is 66.9 degree.

From this figure it is seen that the maximum value $K_P$ for compensated system somewhat increased ($K_P = 7.846$). Thus the good performance of closed loop system is obtained.

![Fig. 12 Stable and unstable regions for first order time delay system with using lead compensator (Example 2)](image)

To obtaining desired values of gain-phase margin tester it is necessary the system phase margin increased to 80 degree.

Doing the procedures provided in section 2 for designing lead compensator, control system was compensated has the bode diagram which is shown in Fig.15. It can be seen from this figure that the phase margin in favourable frequency has reached to desired phase margin (i.e. 80 degree) and the system gain has changed very little. Step responses of closed loop system by using of lead compensator are shown in Fig.16.

![Fig. 13 Stability boundary region for first order time delay system with using lead compensator (Example 2)](image)

![Fig. 14 Bode diagram for control system by $PD^{0.5}$ without using lead compensator (Example 2)](image)

![Fig. 15 Bode diagram for control system by $PD^{0.5}$ without using lead compensator (Example 2)](image)

![Fig. 16 Step responses of closed loop system by using of lead compensator (Example 2)](image)
IV. EXAMPLES COMPARISON

In this section comparison between controlled systems with and without applying lead compensator is exhibited in Table 1. In this table some parameters are compared base on the best step responses. (The best step response for closed loop system with FO-PID is obtained in $K_P = 3$ and with FO-PD is arrived in $K_P = 2.5$.) Based on TABLE I it is seen that increasing lead compensator to controlled system results increase in maximum of $\omega$ and instability boundary for $K_P$. On the other hand overshoot is decreased. Therefore lead compensator can improve performances of controlled systems.

<table>
<thead>
<tr>
<th>Closed loop systems</th>
<th>FO-PID $K_P = 3$</th>
<th>FO-PID With Lead $K_P = 3$</th>
<th>FO-PID $K_P = 2.5$</th>
<th>FO-PID With Lead $K_P = 2.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Settling time</td>
<td>3.5698</td>
<td>3.634</td>
<td>2.36</td>
<td>8.1348</td>
</tr>
<tr>
<td>Instability boundary for $K_P$</td>
<td>8.125</td>
<td>8.34</td>
<td>7.668</td>
<td>7.846</td>
</tr>
<tr>
<td>Overshoot</td>
<td>8.332</td>
<td>0</td>
<td>8.8375</td>
<td>4.2952</td>
</tr>
<tr>
<td>Rise time</td>
<td>5.9871</td>
<td>7.3548</td>
<td>8.7449</td>
<td>6.125</td>
</tr>
<tr>
<td>Rise time</td>
<td>1.6239</td>
<td>3.4368</td>
<td>1.5581</td>
<td>1.8924</td>
</tr>
</tbody>
</table>

V. CONCLUSION

In this paper a class of integrating time delay system using FO-PID and FO-PD controller is examined and stabilization and phase-margin specification are considered, respectively.

Simulation studies have shown that using the FO-PID and FO-PD controllers with lead compensator can achieve bigger closed-loop stability region which are compared to the conventional a class of FO-PID and FO-PD controllers. On the other hand step response of the closed loop system for bigger value of controller gain has been oscillatory. In spite of rise time is increased very few but the overshoot is decreased notably. It is presented FO-PID and FO-PD with lead compensator can better control first order integrating time delay systems.

REFERENCES


