Robust Control Design for Mechanisms with Backlash
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Abstract- A robust feedback control is designed for mechanisms with backlash, where regulation problem is solved under the presence of external perturbation. Appealing the static dead-zone model as a model for backlash, the nonlinear problem is posted in linear $H_\infty$ framework. Experimental realization in an industrial emulator with backlash is carried out, evidencing an acceptable performance.

Keywords– Backlash Compensation; Robust Control; Mechanical Systems; Industrial Emulator

I. INTRODUCTION

In engineering applications, an important nonlinearity that complicates control systems performance is backlash. Backlash can be viewed as “a small gap between a pair of mating gears” (see [1] for a backlash survey). Habitually, a mechanical system with backlash is composed by a driving motor connected to an inertial load via a gear connection. Between this gear connection, the play of moveable parts produces backlash. From the control point of view, there exist many models of backlash compensation [2]–[5]. In basic control literature, a dead-zone model for control design is often used [6]–[9]. This static model is a simplification of diverse physical phenomena with negligible dynamics [10]. In [9], the authors present an application of the dead-zone model of backlash, studying the effect of backlash in electric actuator transmission system, evidencing the goodness of this model. So, applying the dead-zone model for backlash with robust controller, mechanisms with backlash can be controlled with good behavior. Another common approach to control systems with backlash is the adaptive control technique. In [9, 10], an adaptive control law is proposed for stabilizing a closed-loop system that contains a backlash nonlinearity. This control law insures boundedness in closed-loops signals. One of the main drawbacks of this technique is the necessity to rely on an inverse model to compensate the backlash nonlinearity, which contains several parameters that are on line adaptively estimated. This technique depends on the model, which introduces an uncertainty when mechanical systems are considered. Moreover, in practical application where a simple controller is desired, using a nonlinear adaptive controller may not be appropriate. In [15], an admissible nonlinear feedback controller is designed such that the driver load is regulated to a desired position, in the presence of backlash and external disturbance. This controller solves the $H_\infty$ control problem locally. The main disadvantage is that the control algorithm design depends directly on the backlash amplitude, involving difficulties in numerical computation. Moreover, in control realization, this parameter is often unknown and time-varying.

Our objective is to design a dynamic linear $H_\infty$ controller able to stabilize the mechanical system with backlash nonlinearity, under external disturbances. The control design is completed by viewing the nonlinear backlash term as an external perturbation. To fulfill this, the backlash is modeled by a static dead-zone function and viewed as perturbation belonging to the extended $L_2$ space. It is clear that the backlash phenomenon is guaranteed to be bounded in infinity norm, not in $L_2$ norm. So, $H_\infty$ theory is applied for this kind of disturbance [19]. Moreover, the proposed $H_\infty$ controller does not depend directly on the backlash parameter. Performance evaluation of the obtained $H_\infty$ controller is studied in numerical simulations and experiments, where the dynamic control only depends on drive position measurements. An industrial emulator with backlash [16] is used for testing our $H_\infty$ linear controller, showing acceptable performance. This mechatronic system includes two disks (drive and load disks), and a servo actuator with backlash (see Figure 1).

![Mechanical system](image)

**Fig. 1** Mechanical system

The paper is structured as follows. Section II presents the mathematical model of the mechanical system with backlash, modeled by a static dead-zone. Section III presents the control objective, followed by a linear solution to the output feedback $H_\infty$ control problem. Numerical results are studied in Section IV displaying acceptable performance of the proposed controller for several values of backlash amplitudes. In Section V, an application to an industrial emulator with backlash is carried out. Finally, conclusions are stated in Section VI.

II. RELATED WORK

This section shows first the state-space model of the mechanical system with backlash. Then, the control objective
is stated.

**A. Dynamic Mechanical Model**

Consider the motor-load problem (see Figure 2 and [16]) governed by:

\[ J_m \ddot{\theta}_m (t) = -c_m \dot{\theta}_m (t) - T_s + u(t) \]  
\[ J_l \dot{\theta}_l (t) = -c_l \dot{\theta}_l (t) + T_s \]  

Where \( J_m \) (kg m²), \( J_l \) (kg m²), \( c_m \) (Nm/rad/s), \( c_l \) (Nm/rad/s) are the motor inertia moment, load inertia moment, viscous motor friction and the viscous load friction, respectively; \( T_s \) (N·m) is the transmitted shaft torque, and \( u(t) \) is the control input torque; \( \theta_m (t), \theta_l (t) \) are the motor and load angles, respectively (in radians). To define the backlash model, consider the situation illustrated in Figure 3.

When backlash is present in the actuator, the effective control is not \( u(t) \) but \( u_{eff} (t) \) (see Figure 3). The backlash nonlinearity is captured here using the dead-zone model \(^{[1]}\) and modeled by \( D_a (\theta_m - \theta_l) \):

\[ D_a (\theta_m - \theta_l) = \begin{cases} 
\theta_m - \theta_l - \alpha & \text{if } \theta_m - \theta_l > \alpha \\
0 & \text{if } |\theta_m - \theta_l| < \alpha \\
\theta_m - \theta_l + \alpha & \text{if } \theta_m - \theta_l < -\alpha 
\end{cases} \]  

where \( 2\alpha \) is the width of the dead-zone (Figure 2). According with \(^{[1]}\), the dead-zone model for backlash is applicable in (1)-(2) and can be expressed as:

\[ T_s = K_s D_a (\theta_m - \theta_l) \]  

where \( K_s \) (Nm/rad) is the shaft elasticity. Equation (3) can be rewritten as:

\[ D_a (\theta_m - \theta_l) = \theta_m - \theta_l + d_a (\theta_m - \theta_l) \]  

where \( d_a \) is the backlash, which is considered as a perturbation. But the backlash perturbation is guaranteed to be bounded in infinity norm, not in \( L_2 \) norm. As in \(^{[18]}\)-\(^{[19]}\), the truncated norm is considered and the nonlinear problem is posed in linear \( H_{\infty} \) framework \(^{[12]}\), using the extended state space \( L_2 \) to deal with backlash effect. That is, by \(^{[12]}\), if the problem is solved for any disturbance \( w(t) \in L_2 \), then the transfer function satisfies \( \|T_{zw}\|_{\infty} < \gamma + \varepsilon \) for disturbance \( w(t) \in L_2 \), in finite time. The term \( \varepsilon \) depends on Lyapunov function \(^{[11]}\) and vanishes with zero initial conditions.

Therefore, we solve the linear \( H_{\infty} \) problem with the extended perturbation \( w(t) \) = \( d_a (t) n_1 (t) \), where \( n_1 (t) \) is an output additive noise. The next auxiliary model is then considered:

\[ \dot{x}(t) = Ax(t) + B_{10} w(t) + B_2 u(t) \]

\[ \dot{y}(t) = C_2 x(t) + D_{21} w(t) \]

\[ \dot{z}(t) = C_1 x(t) + D_{12} u(t) \]

where \( y(t) \) is a measurable output, \( z(t) \) is a virtual output, and

\[ B_{10} = \begin{bmatrix} 0 & 0 \\ - \frac{k_s}{J_m} & 0 \\ 0 & - \frac{k_s}{J_l} \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

**Remark 1.** It can be shown that the use of the dead-zone model instead of the exact backlash model induces a bounded error \(^{[20]}\). From (6) we can appreciate that \( \| d_a (\cdot) \| \leq \alpha \), so \( d_a \) \in \( L_\infty \) or \( L_2 \). Moreover, from (5), the dead-zone model can be considered as a combination of a linear part that depends on motor and shaft position \( \theta_m - \theta_l \), plus a bounded disturbance \( d_a \).

**B. State-Space Representation of the Mechanical Problem**

From Figures 2 and 3, we see that in the present case, \( u_{eff} (t) = \begin{bmatrix} 0 \\ u(t) - T_s \\ 0 \end{bmatrix}^{[15]} \). Defining the vector space as \( x(t) = \begin{bmatrix} \theta_m (t) \\ \dot{\theta}_m (t) \\ \theta_l (t) \\ \dot{\theta}_l (t) \end{bmatrix} \) the state-space representation of system (1)-(6) yields to

\[ \dot{x}(t) = Ax(t) + B_1 d_a (u(t)) + B_2 u(t) \]  
\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ - \frac{k_s}{J_m} & \frac{k_s}{J_m} & 0 & 0 \\ 0 & 0 & 1 & - \frac{c_l}{J_l} \\ \frac{k_s}{J_l} & - \frac{c_l}{J_l} & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ - \frac{k_s}{J_m} \\ 0 \\ \frac{k_s}{J_l} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]  

The original System (1)-(6) has a nonlinear term due to backlash, which is considered as a perturbation. But the backlash perturbation is guaranteed to be bounded in infinity norm, not in \( L_2 \) norm. As in \(^{[18]}\)-\(^{[19]}\), the truncated norm is considered and the nonlinear problem is posed in linear \( H_{\infty} \) framework \(^{[12]}\), using the extended state space \( L_2 \) to deal with backlash effect. That is, by \(^{[12]}\), if the problem is solved for any disturbance \( w(t) \in L_2 \), then the transfer function satisfies \( \|T_{zw}\|_{\infty} < \gamma + \varepsilon \) for disturbance \( w(t) \in L_2 \), in finite time. The term \( \varepsilon \) depends on Lyapunov function \(^{[11]}\) and vanishes with zero initial conditions.

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where \( 2\alpha \) is the width of the dead-zone (Figure 2). According with \(^{[1]}\), the dead-zone model for backlash is applicable in (1)-(2) and can be expressed as:

\[ T_s = K_s D_a (\theta_m - \theta_l) \]  

where \( K_s \) (Nm/rad) is the shaft elasticity. Equation (3) can be rewritten as:

\[ D_a (\theta_m - \theta_l) = \theta_m - \theta_l + d_a (\theta_m - \theta_l) \]  

where
The matrix $C_2$ will be defined according with the available sensors.

**Remark 2.** Strictly speaking, the nonlinear term $d_\alpha$, which is viewed as perturbation, belongs to the $L_\infty$ space. This justifies the use of linear $H_\infty$ control theory. Moreover, if $n_1$ belongs to $L_2$, then $\theta$ belongs to $L_\infty$ (or $L_\infty \cap L_2$).

### C. Control Objective

Linear $H_\infty$ control problem consists in designing an admissible controller $u(t)$ such that the transfer function $\frac{T_{zw}}{T_z}$ from external disturbance $w(t)$ to measurable output $z(t)$ satisfies $\|T_{zw}\| < \gamma$, where $\gamma$ is a positive constant [13].

The aim of this work is to use this theory to design a robust output feedback controller solving an $H_\infty$ problem when external perturbations present, for a mechanical system with backlash, presented in (9). Two properties have to be verified:

a) Admissibility of the controller $u(t)$,

b) Given the system (9), under $L_2$ perturbation and initialized at the equilibrium point, the following inequality is satisfied [19]:

$$\int_0^T \|z(t)\|_2 dt \leq \gamma \int_0^T \|w(t)\|_2 dt$$

for all $T \geq 0$ and $w \in L_2(0,T)$ (or $L_2$) (see [19] for relationship between linear $H_\infty$ theory and $L_2$ gain).

The System (9) satisfies the standard assumptions in $H_\infty$ control theory [13]. To complete the $H_\infty$ control statement, we propose to design the following output controller $u(t)$:

$$K: \{\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \}

u(t) = C_c x_c(t)$$

also noted as

$$K(s) = \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$$

**Proposition 1.** Under assumptions of linear $H_\infty$ theory, if the closed-loop System (9)-(10) defines an admissible $H_\infty$ controller for $\gamma > 0$, the System (9)-(10) is a locally $H_\infty$ robust system. Moreover, for all $t \in [0,\tau]$ and any $w \in L_2$:

$$\|z_r\|_{L_2} \leq \gamma \|z_r\|_{L_2} + \sqrt{2V(x_0)}$$

If $x_0 = 0$, then $\|z_r\|_{L_2} \leq \gamma \|w\|_{L_2}$.

**Proof.** Straightforward from Theorem 1 [12][13]. If $H_\infty$ conditions hold for a linear system, then $V(x_0) = 0$ and $\|z_r\|_{L_2} \leq \gamma \|w\|_{L_2}$.

### III. SIMULATIONS TO VALIDATE THE CONTROL APPROACH

In this section, numerical simulations are presented with an external disturbance in the motor disk, that is, $y(t) = \theta_m(t) + n_1(t)$, where $n_1(t)$ is an output additive noise. Matrix $C_2$ presented in (9) is defined as:

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad (11)$$

considering only drive position measurements available.

**Case Study.** Given System (9), design a linear output feedback $H_\infty$ Controller (10).

To test the performance of the proposed controller, the following numerical values are considered: $J_m = 0.4$ Kg m$^2$, $J_l = 6$ Kg m$^2$, $c_m = 0.1$ Nm/(rad/s), $c_l = 10.1$ Nm/(rad/s) and $K_e = 3000$ Nm/rad [1]. Employing $\gamma$-iteration and the Matlab Robust Control Toolbox, we obtain the optimal value $\gamma^* = 341.35$. However, to avoid high-gain control, we prefer to keep the sub-optimal value $\gamma = 350$ [2]. With this value of $\gamma$, we obtain the following controller (using notation in (10)):

$$A_c = \begin{bmatrix} -76.77 & 0.00 & 0.00 \\ -10447.12 & -0.77 & 7496.73 & -7.72 \\ 5.12 & 0.00 & 0.00 & 1.00 \\ 696.88 & 0.00 & -500.00 & -0.17 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0.89 \\ 34.02 \\ -0.06 \\ -2.27 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -9.11 & -17.91 & -113.39 & -267.42 \end{bmatrix} \quad (12)$$

To complete numerical experiments, we use the static dead-zone for backlash Modeling (6), with several value of the width of the dead-zone $\alpha$ (3). The next external-perturbation is considered:

$$n_1(t) = \begin{cases} -10 & 8 < t < 14 \\ 20 & 30 < t < 35 \\ 0 & \text{elsewhere} \end{cases} \quad (13)$$

Numerical results are displayed in Figure 4, where the motor and load trajectory are shown. The $H_\infty$ control objective can be interpreted as the ability of the controller to keep the trajectories of the shaft and load movements close to the equilibrium point (the origin of the system), when the system is perturbed and with backlash nonlinearity. In Figure 4, we can see that the angle $\theta_m$ falls in $(-\alpha, \alpha)$ approximately, according to the width of the dead-zone. Also, the backlash effect can be appreciated in the load angle position $\theta_l$, but it remains in a smaller region: $\theta_l \in (-0.2, 0.2)$ in Figure 4. The control effort is pictured in Figure 5, where backlash and external disturbances effects are appreciated. Figures 6.1-6.4 show numerical results for different values of $\alpha$ and under two different external perturbations, evidencing the robustness of the proposed Controller (16) vis-à-vis the backlash uncertainty.

![Fig. 4 Numerical results for $K(s)$ (12), with $\alpha = 2$ rad, zero initial conditions, and perturbation $\gamma(t)$ (13)
Remark 3. The Controller (12) satisfies the $H_\infty$ norm independently of backlash parameter $\alpha$ (this can not be the case, for example, for the controller given in [15]). But from an experimental point of view, a bound of control effort has to be taken into account, limiting the value of $\alpha$.

IV. EXPERIMENTS: APPLICATION TO AN INDUSTRIAL EMULATOR

In this section, a description of the experimental set of an Industrial Emulator is given, along with experimental results.

A. Experimental Setup

Experiments have been performed on an ECP Model 220 industrial emulator that includes a PC-based platform and DC brush-less servo system [16]. The mechatronic system includes two motors, one as servo actuator and the other as disturbance input (not used here), a power amplifier and two encoders which provide accurate position measurements; i.e., 4000 lines per revolution with 4X hardware interpolation giving 16000 counts per revolution to each encoder; 1 count (equivalent to 0.000392 radians or 0.0225 degrees) is the lowest angular measurable [16]. The system was setup to incorporate inertia. The backlash is introduced at the idler pulley SR assembly between drive disk and load disk, by a screw adjustment. In this experiment, the screw is not clamp in order to obtain backlash effect, with approximately $\alpha = 9$ degrees. The drive and load disks were connected via a 4:1 speed reduction (see Figure 1). This reduction is taken into account as uncertainty in the nominal values.

A pentium 4, 2.80GHz, 512 MB RAM computer running under Windows XP is programmed as controller together with the interface medium ECP USR Executive 5.1, a C-like programming language [16]. The system contains a data-acquisition board for digital to analogical conversion and a counter board to read encoder outputs into the servo system. The minimum servo-loop closure sampling time $T_s$ is 0.884 ms. The output voltage generated by the system is in the range of $\pm$ 5 V and is delivered to the motor drive via the DAC. The measurement feedback is a position signal (in counts or radians), measured at the shaft of each of the two disks by the optical rotary incremental position encoders, then it is read by the microcomputer by means of the counter board and delivered into the PC. In the present experiment, the controller only uses position sensor at the motor side. A software interface has been built to easily transfer the raw data collected from the plant (by means of the ECP USR Executive program) to the Matlab workspace environment.

B. Experimental Results

The original experiment is modified adding four masses to the load disk (see Figure 1). Each mass weight is 0.50 kg.
each, located at a radius $r = 10.0\text{cm}$. The resulting calculated reflected inertia to the drive $I_l = 0.0271\text{ kg m}^2$ was used as nominal inertia parameter in $H_{\infty}$ control design. The other parameters were estimated as $J_m = 0.0004\text{ kg m}^2$, $c_m = 0.002\text{ Nm/(Rad/s)}$, $c_l = 0.05\text{ Nm/(Rad/s)}$ and $K_s = 1\text{ Nm/Rad}$. Using the technique presented in Section III for System (9), the next $H_{\infty}$ controller has been obtained:

$$A_c = \begin{bmatrix} -40.56 & 1.00 & 0.00 & 0.00 \\ -4557.63 & -46.65 & 198.72 & -422.18 \\ 0.61 & 0.00 & 0.00 & 1.00 \\ 50.91 & 0.00 & -36.90 & -1.85 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0.57 \\ 11.64 \\ -0.01 \\ -0.20 \end{bmatrix}$$

$$C_c = [-34.93, -1.27, -65.09, -11.94]$$

(14)

This controller only uses drive motor position measurements and its design is $\alpha$-independent. The sub-optimal value of $\gamma$ was estimated as 60. In the experiments, the output controller in (14) was multiplied by 0.5 and the input by 4 to compensate the software gain and the gear gain, respectively. The mechanical system has encoders which give accurate position measurements of the motor and the load shaft; however, our controller employs just the motor shaft position information. In the experimental setup, the backlash amplitude effect was adjusted approximately at $\alpha = 9$ degrees. The experimental results are depicted in Figures 7-10.

Figure 7 displays the control effort supplied to the mechanical system actuator. In our platform, this supplied voltage goes into a power amplifier and, according to the manufacturer $[15]$, it is converted into a torque by multiplication factor of 0.2 Nm/V. In order to prove the robustness of the Controller (14), approximately at times $1s$, $16s$ and $18s$, a perturbation by hand $[20]$ at the load disk has been introduced ($L_2$ disturbances).

![Fig. 7 Control effort](image)

Figure 8 shows the stabilization of the load disk position, by the robust controller in front of the external perturbations. Figure 9 pictures the drive disk position, where a perturbation by hand has been introduced, in order to show the robustness of the proposed dynamic controller. This kind of perturbation is common in robust control experiments $[20]$. Note that after the first perturbation, the drive position stabilizes at the backlash amplitude value. Moreover, it can be appreciate the backlash phenomena also in the load disk, but in a smaller region. Figure 10 pictures the motor angle versus the load disk position, evidencing the backlash hysteretic loop.

![Fig. 8 Load disk position](image)

![Fig. 9 Motor drive disk position](image)

![Fig. 10 Motor angle position versus load angle position](image)

V. CONCLUSIONS

In this paper, we develop a robust output feedback controller for a mechanical system with backlash. The proposed problem was to design an $H_{\infty}$ controller such that the closed-loop system displays an acceptable performance when the system is perturbed under backlash effect. For control design, we have used the static dead-zone model to define backlash, allowing to consider it as an external disturbance. Then, a linear $H_{\infty}$ control theory has been employed to design the controller for $L_2$ disturbance. Moreover, the proposed control design not depends on backlash parameter. Robustness against variation in backlash parameter is guaranteed in terms of $H_{\infty}$ objectives. An Industrial Emulator with backlash, which is a mechanical system used for control performance evaluation, was configured to examine the performance of the proposed $H_{\infty}$ control technique, showing an acceptable achievement when only drive position measurement is employed.
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